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ON THE EXISTENCE OF WEIERSTRASS GAP SEQUENCES ON TRIGONAL CURVES*

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We find a family of sequences which contains all possible Weierstrass nongap sequences at non-ramification points of the trigonal covering map. And we prove by constructing various examples that each sequence in this family is really the Weierstrass nongap sequence at a non-ramification point on some trigonal curve.

1. Introduction

Let C be a nonsingular complex projective algebraic curve (or a compact Riemann surface) of genus g. Let $\mathcal{M}(C)$ denote the field of meromorphic functions on C. For a point $p \in C$, we define the *Weierstrass nongap sequence* $H_p \subset \mathbb{N}$ by

$$H_p = \{n \mid \text{there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = np\},$$

and the *Weierstrass gap sequence* $G_p \subset \mathbb{N}$ by

$$G_p = \mathbb{N} - H_p = \{n \mid \text{there is no } f \in \mathcal{M}(C) \text{ with } (f)_\infty = np\}.$$

It is well known that H_p is a sub-semigroup of $\mathbb{N}$ and

$$\operatorname{card} G_p = g.$$

A point $p \in C$ is called a *Weierstrass point* if $G_p \neq \{1, 2, 3, \dots, g\}$.

Since the nature of a Weierstrass point is closely related to the behaviour of meromorphic functions on a given curve, it is natural to look at linear series on a curve. We denote by g_n^r a complete linear series of degree n, and of dimension r. And if $D \in g_n^r$, we sometimes use the notation $|D|$ instead of g_n^r. We say that the curve C is *trigonal* if there is a rational map $\phi : C \to \mathbb{P}^1$ whose degree is three, or equivalently there exists a base point free g_3^1 on C.

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We say that g_n^r is compounded of g_k^1 if

$$g_n^r = rg_k^1 + B,$$

where B is the base locus of g_n^r.

In this paper we exclusively deal with trigonal curves.

2. The sequences which can be Weierstrass gap sequences on a trigonal curve

Recently Coppens, Kato and Horiuchi obtained the following result [2, 3, 5]:

Theorem. *Let C be a trigonal curve and $\phi: C \to \mathbb{P}^1$ be a three sheeted covering map. If p is a ramification point of ϕ, then G_p is one of the following:*

$$\{1,2,4,5,\dots,3n-2,3n-1,3n+1,3n+4,\dots,3(g-n-1)+1\},$$

$$\{1,2,4,5,\dots,3n-2,3n-1,3n+2,3n+5,\dots,3(g-n-1)+2\},$$

$$\{1,2,3,\dots,2n-1,2n,2n+1,2n+3,\dots,2g-2n-1\},$$

$$\{1,2,3,\dots,2n-1,2n,2n+2,2n+4,\dots,2g-2n\},$$

where $(g-1)/3 \le n \le g/2$. Conversely, each of these sequences actually occurs as G_p for some trigonal curve C and $p \in C$.

Proof. See [2, 3, 5]. □

Now we are interested in a point on a trigonal curve which is not a ramification point of the trigonal covering ϕ. The following lemmas turn out to be useful for the investigation of the matter:

Lemma 2.1. *Suppose g_n^r and $g_{n'}^{r'}$ are complementary linear series on a trigonal curve, i.e., $|g_n^r + g_{n'}^{r'}| = K$, where K is the canonical series. Then one of these is compounded of g_3^1.*

Proof. See [1, p. 426]. □

Lemma 2.2. *Assume that*

$$g_n^r + g_{n'}^{r'} = K.$$

If p is a base point of g_n^r, then p is not a base point of $|g_{n'}^{r'} + p|$, hence

$$\dim |g_{n'}^{r'} + p| = r' + 1.$$

Proof. This follows easily from the Riemann–Roch theorem. □

Here we introduce a new parameter which plays an important role throughout this paper. For $p \in C$, *define* $s(p)$ by

$$s(p) = \max\{m \in \mathbb{N} \mid mp \text{ is special}\},$$

and if there is no danger of confusion we abbreviate $s(p)$ to s.

Lemma 2.3. *The following hold*:
(1) $g-1 \leq s(p) \leq 2g-2$.
(2) $\dim |s(p)p| = s(p) - g + 1$.

Proof. The proof of (1) is obvious. By the maximality of $s(p)$,

$$\dim |K - s(p)p| = 0,$$

hence by the Riemann-Roch theorem,

$$\dim |s(p)p| = s(p) - g + 1. \qquad \square$$

Remark 2.4. Note that $s(p) = g-1$ if and only if p is not a Weierstrass point.

Lemma 2.5. *The set* $\{k \in \mathbb{N} \mid k \geq s(p)+2\}$ *is contained in* H_p *but* $s(p)+1$ *is not an element of* H_p.

Proof. Since $(s(p)+1)p$ is nonspecial, by the Riemann-Roch theorem

$$\dim |(s(p)+1)p| = s(p) - g + 1.$$

Comparing with Lemma 2.3,

$$\dim |(s(p)+1)p| = \dim |s(p)p|,$$

which implies that $s(p)+1 \notin H_p$. For $k \geq s(p)+2$, $(k-1)p$ is nonspecial and hence

$$\dim |kp| = k - g = \dim |(k-1)p| + 1.$$

Thus $|kp|$ is base point free or equivalently k is a nongap. $\square$

With this preparation, we are now ready to prove

Theorem 2.6. *Let C be a trigonal curve of genus $g \geq 5$. Let $p \in C$ be a non-ramification point but a Weierstrass point on C. Then the Weierstrass nongap sequence at p is of the form*

$$a, a+1, a+2, \ldots, a+(s-g), s+2, s+3, \ldots,$$

for some a such that $g \geq a \geq [(s+1)/2]+1$, where [ ] *is the Gauss symbol.*

Proof. Since p is a Weierstrass point, by Remark 2.4, $s \geq g$. By the definition of

$s=s(p)$, there exists a canonical divisor of the form

$$sp+p_1+p_2+\cdots+p_{2g-2-s},$$

where $p_i\in C$, $p_i\neq p$ for $i=1,2,\ldots,2g-2-s$. Since p is not a ramification point, the first nongap a is greater than 3, hence $|ap|$ is not compounded of g_3^1. Then by Lemma 2.1, the residual series to $|ap|$, i.e.,

$$|(s-a)p+p_1+\cdots+p_{2g-2-s}|$$

is compounded of g_3^1. Let D be the divisor $(s-a)p+p_1+\cdots+p_{2g-2-s}$. By the Riemann-Roch Theorem, dim $|D|=g-a$, hence

$$|D|=(g-a)g_3^1+B,$$

where B is the base locus of $|D|$. Thus each divisor in $|D|$ is of the form

$$T_1+\cdots+T_{g-a}+B,$$

where $T_i\in g_3^1$ for $i=1,2,\ldots,g-a$. Let T be the element in g_3^1 such that $T\geq p$. Then $T=p+q_1+q_2$ for some $q_1,q_2\in C$, where q_1 and q_2 are distinct from p since p is not a ramification point. Now by the maximality of $s=s(p)$,

$$D=(g-a)T+B=(g-a)(p+q_1+q_2)+B=(s-a)p+p_1+\cdots+p_{2g-2-s}.$$

Thus

$$p_1+\cdots+p_{2g-2-s}\geq(g-a)q_1+(g-a)q_2.$$

Hence

$$2(g-a)\leq 2g-2-s$$

or

$$a\geq(s+2)/2$$

or

$$a\geq[(s+1)/2]+1.$$

On the other hand,

$$B\geq\{(s-a)-(g-a)\}p=(s-g)p,$$

in other words, $(s-g)p$ is contained in the base locus of $|D|$ which is residual to $|ap|$. Therefore by Lemma 2.2,

$$a,a+1,\ldots,a+(s-g)$$

are nongaps at p.

Now combining with Lemma 2.5, the set

$$S=\{a,a+1,a+2,\ldots,a+(s-g),s+2,s+3,\ldots\}$$

is contained in H_p, and the cardinality of $\mathbb{N}-S$ is exactly g. Thus the proof is complete. □

3. Existence of a point and a trigonal curve with a nongap sequence in Theorem 2.6

In this section, we show that each sequence in Theorem 2.6 actually occurs as a Weierstrass nongap sequence at some point on a trigonal curve, by constructing examples.

For a triple of integers (α, j, i) satisfying

$$2 \leq \alpha \leq 3, \qquad 3 \leq j \leq d-1, \qquad j-2 \leq i \leq d-3,$$

consider a linear system of polynomials of degree d

$$\Lambda = \{F_\lambda(x, y, 1)\colon \lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{P}^4\},$$

where

$$F_\lambda(x, y, 1) = \lambda_0 y^3 x^{d-3} + \lambda_1 y^\alpha x^i + \lambda_2(y^3 - y^2 x^{d-j}) + \lambda_3(yx^{j-1} - x^{d-1}) + \lambda_4(y - x^{d-j}).$$

Since the common base loci are the three points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, by Bertini's theorem a generic element of this linear system is smooth except for these three points. But neither $(1,0,0)$ nor $(0,0,1)$ is a singular point of F_λ if $\lambda_3 \neq 0$ and $\lambda_4 \neq 0$. Choose an element F of Λ satisfying the following:

(a) $\lambda_0 = 1$, $\lambda_1 \neq 0, -1$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$, $\lambda_4 \neq 0$.

(b) F is smooth except for $(0,1,0)$.

(c) $x^{d-3} + \lambda_1 x^i + \lambda_2$ and $(d-3)x^{d-4} + \lambda_1 i x^{i-1}$ do not have a common root.

Lemma 3.1. *F satisfies the following*:

(1) *F has an ordinary singularity at $(0,1,0)$ of multiplicity $d-3$, and $C = Z(F)$ is a trigonal curve of genus $2d-5$.*

(2) *Let $G(x, y, 1) = y - x^{d-j}$, $p = (0,0,1)$. Then the intersection multiplicity of F and G at p is*

$$I(F \cap G, p) = \alpha(d-j) + i.$$

(3) *$s(p) = \alpha(d-j) + i + (j-3)$ and for each j,*

$$g \leq s(p) \leq (2g-2) - 2(j-3).$$

(4) *The nongap sequence at p is*

$$g-(j-3) \to s-(j-3),\ s+2 \to,$$

where "$\to$" means all consecutive integers in the range.

Proof. (1) Since the degree of F with respect to y is 3, F has a singularity at $(0,1,0)$ of multiplicity $d-3$. Moreover, the coefficient of y^3 is

$$\begin{cases} x^{d-3} + \lambda_2 & \text{if } \alpha = 2, \\ x^{d-3} + \lambda_1 x^i + \lambda_2 & \text{if } \alpha = 3. \end{cases}$$

Hence both have distinct roots by condition (c) of F. Thus $(0,1,0)$ is an ordinary singularity. By the choice of F, we conclude that F has only one singularity, which is an ordinary one of multiplicity $d-3$. By Bezout's theorem, F must be irreducible. Then by the genus formula for a plane curve with ordinary singularities, the genus of C is $2d-5$. And the system of lines through $(0,1,0)$ cuts out g_3^1, i.e., C is trigonal.

(2) Since the last three terms of F are multiples of $y-x^{d-j}$, we have

$$I(F\cap G, p)=\alpha(d-j)+i.$$

(3) It is well known that the curves of degree $d-3$ which have a singularity at $(0,1,0)$ of multiplicity $d-4$ cut out the canonical series on $C=Z(F)$, and that those curves have the equations of the form

$$yL_1(1,x,x^2,\ldots,x^{d-4})+L_2(1,x,x^2,\ldots,x^{d-3}),$$

where L_1 and L_2 are linear forms with respect to their variables. We call these curves special adjoints. For a special adjoint, $G_1=(G)(x^{j-3})$, where G is given in (2), we obtain

$$I(F\cap G_1, p)=\alpha(d-j)+i+(j-3).$$

Now we show that this number is $s(p)$, i.e., the maximal number of the set $\{I(F\cap H, p)\mid H \text{ is a special adjoint}\}$.

Suppose that $\max\{I(F\cap H,p)\mid H \text{ is a special adjoint}\}$ is attained by H_1. If y divides H_1, then H_1 must be of the form yx^{d-4}. Then

$$\begin{aligned}I(F\cap H_1, p)&=I(F\cap (y), p)+I(F\cap (x^{d-4}), p)\\&=(d-j)+(d-4)\\&<(d-j)+d-3+i+(\alpha-2)(d-j)\\&=\alpha(d-j)+i+(j-3)\\&=I(F\cap G_1, p).\end{aligned}$$

This is contradictory to the choice of H_1. Thus H_1 can be written as

$$H_1=x^kH(x,y,1),$$

where $k\geq 0$ and $H(x,y,1)$ is not a multiple of x or y. If $H(x,y,1)$ is constant, i.e., $k=d-3$, then we have a contradiction again. And since $H(0,0,1)=0$ by the choice of H_1, H has the term y and x^n for some $n=1,2,\ldots,d-k-3$. If H has a term x^n for $n<d-j$, then

$$\begin{aligned}I(F\cap H_1, p)&=I(F\cap (x^k), p)+I(F\cap H, p)\\&\leq k+n\\&\leq (d-4)+(d-j)\\&<I(F\cap G_1, p)\end{aligned}$$

as above, and this is also contradictory to the choice of H_1. Thus the degree of H with respect to x and y must be greater than or equal to $d-j$, i.e., $d-k-3 \geq d-j$, and $k \leq j-3$. Since G and H have a singularity at $(0,1,0)$ of multiplicity $d-j-1$ and $d-k-4$ respectively, we obtain

$$I(G \cap H, (0,1,0)) \geq (d-j-1)(d-k-4).$$

On the other hand, since

$$I(F \cap H, p) \geq I(F \cap G, p) = \alpha(d-j)+i,$$

we have

$$I(G \cap H, p) \geq \alpha(d-j)+i.$$

Thus

$$\begin{aligned}\sum_{z \in G \cap H} I(G \cap H, z) &\geq (d-j-1)(d-k-4)+\alpha(d-j)+i \\ &\geq (d-j)(d-k-3)-(d-k-4)+(\alpha-1)(d-j)+(j-2) \\ &> (d-j)(d-k-3).\end{aligned}$$

Note that we use the condition $j-2 \leq i \leq d-3$ for the second inequality. By Bezout's Theorem, we conclude that G and H have a common component. Since G is irreducible, H is a multiple of G. If H/G is not constant, H/G is divisible by x because H/G cannot have the term y and H/G must be zero at p by the choice of H_1. But this is contradictory to the fact that H is not a multiple of x. Thus H/G is constant and $s(p) = I(F \cap G, p) = \alpha(d-j)+i+(j-3)$.

The inequality $g \leq s(p) \leq (2g-2)-2(j-3)$ follows easily from the condition of the triple (α, j, i).

(4) Since $K-sp \geq (j-3)(q_1+q_2)$, where $p+q_1+q_2$ is the divisor cut out by the line $Z(x)$, the numbers $s, s-1, \ldots,$ and $s-(j-4)$ are gaps. And $K-ap$ is compounded of g_3^1 and $K-sp \not\geq (j-2)(q_1+q_2)$, hence $s-(j-3)$ is a nongap. Now the sequence follows from Theorem 2.6. □

Before we prove the main result of this section, we need another lemma which is similar to Lemma 3.1. For a triple of integers (α, j, i) satisfying

$$2 \leq \alpha \leq 3, \qquad 3 \leq j \leq d-1, \qquad j-2 \leq i \leq d-3,$$

consider the linear system of polynomials of degree d

$$\hat{\Lambda} = \{\hat{F}_\lambda(x,y,1) \mid \lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{P}^4\},$$

where

$$\begin{aligned}\hat{F}_\lambda(x,y,1) = \lambda_0 y^3 x^{d-3} &+ \lambda_1 y^\alpha x^i + \lambda_2(y^3 - y^2 x^{d-j}) \\ &+ \lambda_3(yx^{j-1} - x^{d-1}) + \lambda_4(xy - x^{d-j+1}).\end{aligned}$$

Since the common base loci are the three points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, by Bertini's theorem a generic element of this linear system is smooth except for these three points. But $(1,0,0)$ is not a singular point of $\hat{F}_\lambda$ if $\lambda_3 \neq 0$. Choose an element

$\hat{F}$ of $\hat{\Lambda}$ satisfying the following:

(a) $\lambda_0=1,\ \lambda_1\neq 0,-1,\ \lambda_2\neq 0,\ \lambda_3\neq 0,\ \lambda_4\neq 0$.

(b) $\hat{F}$ is smooth except for $(0,1,0)$, and $(0,0,1)$.

(c) $x^{d-3}+\lambda_1 x^i+\lambda_2$ and $(d-3)x^{d-4}+\lambda_1 i x^{i-1}$ do not have a common root.

Lemma 3.2. *$\hat{F}$ satisfies the following:*

(1) *$\hat{F}$ has ordinary singularities at $(0,1,0)$ and $(0,0,1)$ of multiplicity $d-3$ and 2, respectively, and $C=Z(\hat{F})$ is a trigonal curve of genus $2d-6$.*

(2) *Let $G(x,y,1)=y-x^{d-j}$, $\bar{p}=(0,0,1)$. Then the intersection multiplicity of $\hat{F}$ and G at $\bar{p}$ is*

$$I(\hat{F}\cap G,\bar{p})=\alpha(d-j)+i.$$

(3) *Let p be the point on the normalization of the curve $\hat{F}$ corresponding to the tangent line $Z(y)$ at $\bar{p}=(0,0,1)$. Then $s(p)=\alpha(d-j)+i-2+(j-3)$ and $g-1\leq s(p)\leq (2g-2)-2(j-3)$. The second equality holds when $\alpha=2$ and $i=j-2$, and in this case p is not a Weierstrass point.*

(4) *The nongap sequence at p is*

$$g-(j-3)\rightarrow s-(j-3), s+2\rightarrow .$$

Proof. The proofs of (1), (2) and (4) are similar as those of Lemma 3.1, (1), (2) and (4), so we prove only (3).

Let $G_1=(G)(x^{j-3})$. Note that $p+2q\in g_3^1$ is cut out by the line $Z(x)$, where q is the point on the normalization of the curve $\hat{F}$ corresponding to the tangent line $Z(x)$ at $\bar{p}=(0,0,1)$. Thus

$$I(\hat{F}\cap G_1,\bar{p})=\alpha(d-j)+i+3(j-3).$$

Using [4, Proposition 8, p. 207], we conclude that

$$\begin{aligned} s(p)&\geq \alpha(d-j)+i+3(j-3)-2(j-3)-2\\ &=\alpha(d-j)+i+(j-3)-2. \end{aligned}$$

Suppose that $s(p)$ is attained by a special adjoint H_1. If y divides H_1, then H_1 must be of the form yx^{d-4}. Then the multiplicity at p of the canonical divisor $\mathrm{div}(H_1)-E$ (see the notation in [4, p. 207]) is

$$(d-j+1)+(d-4)-1=(d-j)+(d-4)<\alpha(d-j)+i+(j-3).$$

This is contradictory to the choice of H_1. Thus H_1 can be written as

$$H_1=x^k H(x,y,1),$$

where $k\geq 0$ and $H(x,y,1)$ is not a multiple of x or y. If $H(x,y,1)$ is constant, i.e., $k=d-3$, then we have a contradiction again. Also by the choice of H_1, $H(0,0,1)=0$, hence H has the term y and x^n for some $n=1,2,\ldots,d-k-3$. Suppose that H has a term x^n for $n<d-j$. Then the multiplicity at p in the canonical divisor $\mathrm{div}(H_1)-E$

is $k+n$, which is less than $\alpha(d-j)+i+(j-3)-2$. Thus the degree of H with respect to x and y must be greater than or equal to $d-j$, i.e., $d-k-3 \geq d-j$. Now the equality $G=H$ (up to constant) follows from the same computation as in the last part of proof of Lemma 3.1(3). □

Finally, we prove

Theorem 3.3. *If $H \subset \mathbb{N}$ is the sequence of the form in Theorem 2.6, then there exist a trigonal curve C and a point $p \in C$ such that $H_p = H$.*

Proof. We use Lemma 3.1 if g is odd and we use Lemma 3.2 if g is even.

In Lemma 3.1(3) [resp. Lemma 3.2(3)], varying α and i, the number

$$\alpha(d-j)+i+(j-3) \qquad [\text{resp. } \alpha(d-j)+i-2+(j-3)]$$

admits every integer in $\{g, g+1, \ldots, (2g-2)-2(j-3)\}$, because

$$\alpha(d-j)+i+(j-3)=\begin{cases} 3d-j-6 & \text{if } \alpha=2 \text{ and } i=d-3, \\ 3d-j-5 & \text{if } \alpha=3 \text{ and } i=j-2 \end{cases}$$

$$\left[\text{resp. } \alpha(d-j)+i-2+(j-3)=\begin{cases} 3d-j-8 & \text{if } \alpha=2 \text{ and } i=d-3, \\ 3d-j-7 & \text{if } \alpha=3 \text{ and } i=j-2 \end{cases}\right].$$

For each s, $g \leq s \leq 2g-2$, let

$$j_0=\max\{j \mid s \leq (2g-2)-2(j-3)\}.$$

Then $s=((2g-2)-2(j_0-3))$ or $s=((2g-2)-2(j_0-3))-1$. By Lemma 3.1(3),(4) [resp. Lemma 3.2(3),(4)], for each $j \leq j_0$, we obtain a curve and a point on it with Weierstrass nongap sequence

$$g-(j-3) \rightarrow s-(j-3), s+2 \rightarrow .$$

Note that

$$s-(j-3)=g-(j-3)+(s-g)$$

in the above sequence.

Furthermore, the least possible first nongap is

$$g-(j_0-3)=[(s+1)/2]+1,$$

which appeared in Theorem 2.6. □

Remark. By Theorem 2.6 and Theorem 3.3, together with results of Coppens, Kato and Horiuchi, we have determined all the possible Weierstrass nongap sequences on trigonal curves.

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